Chapter 1 Intro What are the differences between Pre-Calc & Calculus?

→ Pre-Calc gave us a basis for more general topics like trigonometry, vectors, and Polar-graphing; Calculus is focused on continuous or instantaneous change

Pre-Calc
→ Areas of basic shapes

Calc
→ Changes in Rates
→ Integral – Area of irregular shapes

Aug. 28. 2024

What is Calculus?

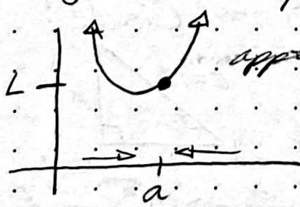
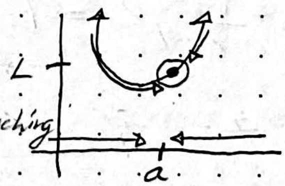
→ Branch of Math, deals with finding derivatives & integrals of functions
↳ Based on methods of summation of infinitesimal differences (smaller & smaller)

2 Branches
→ Differential → slopes, rates of change, derivatives, etc.
→ Integral → areas, volumes, integrals, etc.

Limits are used in both branches

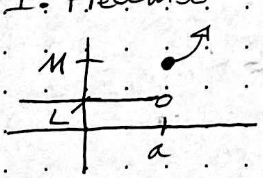
1.2 Finding Limits Graphically

Limit – y-value the function tends to approach from ~~both~~ sides of the x-axis

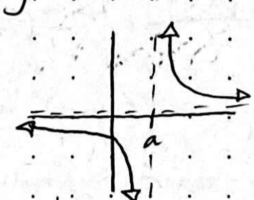
Ex: 
approaching L → $\lim_{x \to a} f(x) = L$
↳ limit notation is true for approaching the limit from both sides x-axis


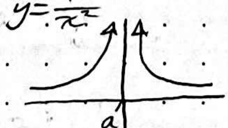
When does a limit not exist?

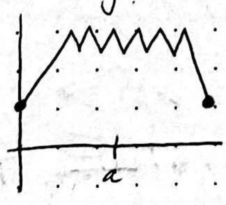
DNE = does not exist
↳ Must state a reason

1. Piecewise

$\lim_{x \to a} f(x) = DNE$
$\lim_{x \to a^+} f(x) = M$
$\lim_{x \to a^-} f(x) = L$
$\neq$

only with jump discontinuity

2. Unbounded behavior
$y = \frac{1}{x-a}$
$\lim_{x \to a} f(x) = DNE$, because of unbounded behavior

$\lim_{x \to a} f(x) = DNE$, because $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$
∞ $-\infty$

$y = \frac{1}{x^2}$

$\lim_{x \to} f(x) = DNE$ due to unbounded behavior
$\lim_{x \to x} f(x) = \infty$

3. Oscillating Behavior

$\lim_{x \to a} f(x) = DNE$ due to oscillation

How to find a limit: (methods)

1. Direct substitution → plug in x to f(x).
2. Use factoring
3. Use rationalization techniques (conjugates)
*4. Use graphing
*5. Use a table

* = When using calculator

Ex 1: Evaluate $f(x) = \frac{x}{(\sqrt{x+1}-1)}$ @ several x-values near 0 & estimate limit

$\rightarrow \frac{0}{(\sqrt{0+1}-1)} \rightarrow \frac{0}{0} \rightarrow$ DNE, cannot divide by 0; indeterminate

Table:

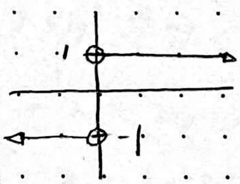
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.9487	1.995	1.9995	ind	2.0005	2.005	2.0[illegible]

$\lim_{x \to 0} f(x) = 2$

Ex. 2: Find limit of $f(x)$ as x approaches $\quad f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$

$\lim_{x \to 2} f(x) = 1$ $\quad f(2) = 0$

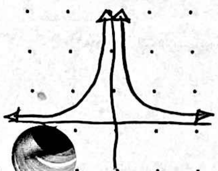
Ex 3: Show that limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist



$\lim_{x \to 0^+} \frac{|x|}{x} = 1$ $\quad \lim_{x \to 0} \frac{|x|}{x} =$ DNE, because $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^-} \frac{|x|}{x} = -1$

Ex 4: Limit $\lim_{x \to 0} \frac{1}{x^2}$ $\quad \lim_{x \to 0} \frac{1}{x^2} = \infty$



$\lim_{x \to 0} \frac{1}{x^2} =$ DNE, due to unbounded behavior

HW: p. 55 Qs: 3, 7, 11-19, 21-23, 49-52, 59.

1.3 Warm-Up

$\lim_{x \to 3} \frac{x-3}{x^2-9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1695	0.1689	0.1667	?	0.1666	0.1664	0.1639

$\lim_{x \to 3} \frac{x-3}{x^2-9} = 0.1666$ $\rightarrow$ the limit exists

3 acceptable reasons for DNE: ① $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

$\frac{n}{0} \rightarrow$ undefined $\quad$ ② Oscillating behavior

$\frac{0}{0} \rightarrow$ indeterminate $\quad$ ③ Unbounded/Boundless behaviour

~~1.3 HW p.67 Qs: 3, 14, 15, 23, 28, 31, 37-45 odd, 51, 55, 57, 59, 63, 64, 73, 85, 86~~

1.3 Eval. limits analytically Aug 27, 2024

$\lim_{x \to c} f(x)$ does not depend on value of f at $x=c$

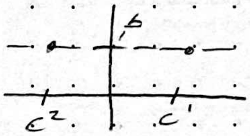
Sometime $f(c)$ is

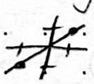
Use substitution first for any limit problem involving a function

Theorem 1.1 Basic Limits

let b & c be real numbers & n a positive integer.

1. $\lim_{x \to c} b = b$ 2. $\lim_{x \to c} x = c$ 3. $\lim_{x \to c} x^n = c^n$


Identity function


Direct substitution

Ex 1: a: $\lim_{x \to 4} 5 = 5$ b: $\lim_{x \to -2} x = -2$ c. $\lim_{x \to 3} x^2 = 9$

Theorem 1.2 Properties of Limits

b & c real numbers, n positive integer, f & g functions with these

limits: $\lim_{x \to c} f(x) = L$ $\lim_{x \to c} g(x) = K$ $b \underbrace{\lim_{x \to c} f(x)}_{L} \to bL$

1. Scalar multiple $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product $\lim_{x \to c} [f(x) g(x)] = LK$
4. Quotient $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ if $K \neq 0$
5. Power $\lim_{x \to c} [f(x)]^n = L^n$

Ex 2: Use properties to evaluate following:

a) $\lim_{x \to 2} (3x^2 - 1) \to 3 \lim_{x \to 2} x^2 - \lim_{x \to 2} 1 \to 3 \cdot 4 - 1 \to \boxed{11}$

Use direct substitution for all kinds of functions.

Ex 3: $\lim_{x \to 1} \frac{x^2 + x + 2}{x+1} \to \frac{1+1+2}{1+1} \to \frac{4}{2} \to \boxed{2}$

Ex 4: ⓐ $\lim_{x \to 0} \sqrt{x^2 + 4}$

$\to \sqrt{0^2 + 4}$

$\to \sqrt{4}$

$\to 2$

ⓑ $\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$

$\to \sqrt[3]{2(3^2) - 10}$

$\to \sqrt[3]{2(3^2) - 10}$

$\to \sqrt[3]{8}$

$\to 2$

Ex 5:

(a) $\lim_{x \to 0} \tan x \to \tan 0 \to \frac{0}{1} \to \boxed{0}$

(b) $\lim_{x \to \pi} (x \cos \pi) \to \pi \cdot \cos \pi \to \pi(-1) \to \boxed{-\pi}$

(c) $\lim_{x \to 0} \sin^2 x \to (\sin 0)(\sin 0) \to 0 \cdot 0 \to \boxed{0}$

Ex 6:

$\lim_{x \to 1} \frac{x^3-1}{x-1} \to \frac{1^3-1}{1-1} \to \frac{0}{0} \to$ undetermined / indeterminate

$\lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} \to \lim_{x \to 1} (x^2+x+1)$

$$\boxed{\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2) \\ a^3+b^3 &= (a+b)(a^2-ab+b^2) \end{aligned}}$$

$\to 1^2+1+1 \to \boxed{3}$

Ex 7:

$\lim_{x \to -3} \frac{x^2+x-6}{x+3} \to \frac{9+3-6}{-3+3} \to \frac{0}{0} =$ indeterminate.

$\lim_{x \to -3} \frac{(x+3)(x-2)}{x+3} \to \lim_{x \to -3} x-2 \to -3-2 \to \boxed{-5}$

$\to$ Synthetic division

$$\begin{array}{r|rrr} -3 & 1 & 1 & -6 \\ & \downarrow & -3 & 6 \\ \hline & 1 & -2 & 0 \end{array} \to (x-2)$$

Ex 8: $\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} \to \frac{\sqrt{0+1}-1}{0} \to \frac{0}{0}$ indeterminate

$= \lim_{x \to 0} \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) \to x+1 \cancel{-\sqrt{x+1}+\sqrt{x+1}} -1 \to \frac{x}{x(\sqrt{x+1}+1)}$

$\to \frac{1}{\sqrt{x+1}+1} \to \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} \to \frac{1}{\sqrt{1}+1} \to \boxed{\frac{1}{2}}$

Theorem 1.8. Squeeze Theorem / Sandwich Theorem

If $h(x) \leq f(x) \leq g(x)$ in interval containing c, exept c itself,

and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} f(x)$ exists & equals L

Two Special Trig. Limits

$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \& \quad \lim_{x \to 0} \frac{1-\cos x}{x} = 0$

$\frac{\sin a(x)}{a(x)} = 1$

Ex 9: $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{x}\right) \to \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$

$\downarrow$

$\left(\frac{1}{1}\right)(1) \to \boxed{1}$

Learn: Intercepts, domain, solving for x with trig, trinomial factoring
solving triangles trig

1.3 HW p.67 Q 3, 14, 15, 23, 28, 31, 37-45 odd, 51, 55, 57, 59, 63, 64, 73, 85, 86

1.4 Warm up (a) $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} \to \left(\frac{(\sqrt{x+3}-2)}{(x-1)}\right)\left(\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right)$

$\to \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \to \frac{x-1}{(x-1)(\sqrt{x+3}+2)} \to \frac{1}{\sqrt{x+3}+2} \to \frac{1}{\sqrt{1+3}+2} \to \boxed{\frac{1}{4}}$

(b) $\lim_{x\to c} f(x) = 3 \quad \lim_{x\to c} g(x) = 2 \quad$ a) 10 b) 5 c) 6 d) $\frac{3}{2}$

$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \to$ (85) $f(x) = x^2-4x$

$\to \lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-4(x+\Delta x)-(x^2-4x)}{\Delta x} \to \lim_{\Delta x\to 0} \frac{\cancel{x^2}+2x\Delta x+\Delta x^2\cancel{-4x}-4\Delta x\cancel{-x^2}\cancel{+4x}}{\Delta x}$

$\to \lim_{\Delta x\to 0} \frac{\Delta x(2x+\Delta x-4)}{\Delta x} \to 2x+\Delta x-4 \to 2x+0-4 \to \boxed{2x-4}$

(86)
$f(x)=\sqrt{x} \quad \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \to f(x) = \lim_{\Delta x\to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$

$\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\left(\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\cancel{\Delta x}\downarrow}\right) \to \frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x(\cdots)} \to \frac{\Delta x}{\Delta x\sqrt{x+\Delta x}+\sqrt{x}}$

$\to \frac{1}{\sqrt{\Delta x+x}+\sqrt{x}} \to \frac{1}{2\sqrt{x}}$

(64) $\lim_{x\to 0} \frac{3(1-\cos x)}{x} \to 3\left(\lim_{x\to 0}\frac{1-\cos x}{x}\right) \to \frac{1-\cos x}{x} = 0\cdot 3 = \boxed{0}$

1.4 Continuity & 1 directional limits

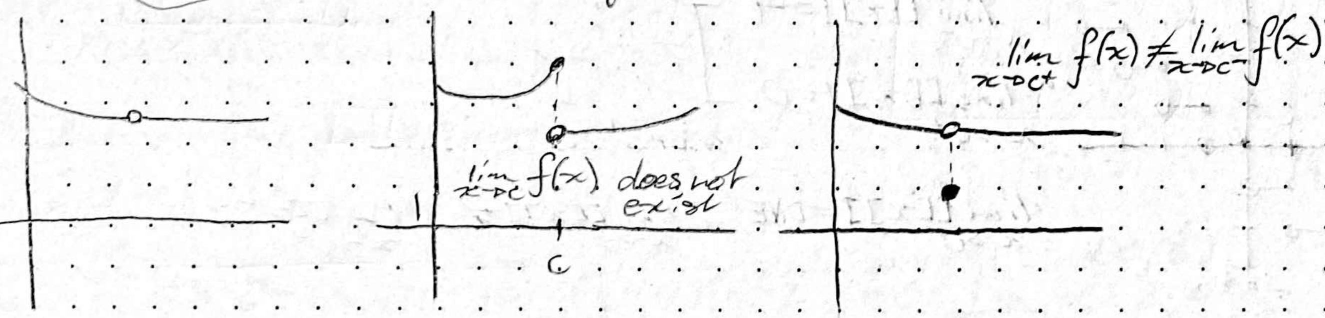
A function is continuous if:

→ there are no jumps → It is always defined

→ there is no hole → there are no vertical asymptotes

$$\frac{(x+3)}{(x+3)(x-3)}$$ Hole

→ Vertical asymptote



Continuity at a point:

Function f is continuous at c if:

1 $f(c)$ is defined

2. $\lim_{x \to c} f(x)$ exists

3. $\lim_{x \to c} f(x) = f(c)$

at an open interval:

if continuous at each point $(-\infty, \infty)$
everywhere continuous

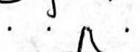
2 types of discontinuity

1. Non-removable → jump & ver. asym. (infinite disc.)
2. Removable → Hole

} if you don't know, use Non-removable discontinuity

Ex:

ⓐ $f(x) = \frac{1}{x}$ $x \neq 0$



$f(x)$ is continuous on $(-\infty, \infty)$, except at $x=0$, where there is a non-removable infinite discontinuity.

ⓑ $g(x) = \frac{x^2-1}{x-1}$

$g(x)$ is continuous on $(-\infty, \infty)$, except at $x=1$, where there is a removable discontinuity

ⓒ $h(x) = \begin{cases} x+1, & x \leq 0 \\ x^2+1, & x > 0 \end{cases}$ $\underbrace{\lim_{x \to 0^-} f(x) = 1 \quad \lim_{x \to 0^+} f(x) = 1}_{\text{equal limits}}$ $h(1) = 1$

$h(x)$ is continuous on $(-\infty, \infty)$

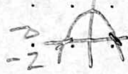
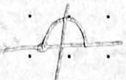
ⓓ $y = \sin x$
The function $y = \sin x$ is everywhere continuous.

One-sided limits → they exist (no DNE)
from right → +
from left → −

$\lim_{x \to 0^+}$ → limit as x approaches from the right

∞ or $-\infty$ work as answers, never DNE

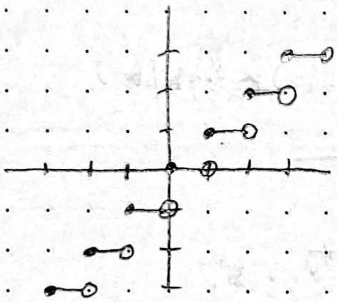
Ex 2: $f(x)=\sqrt{4-x^2}$ $\lim_{x\to -2^+} f(x)=0$

$4-x^2 \to$  $f(x) \to$ 

Ex 3: Greatest Integer Function $\to$ Biggest Integer $\leq x$ in. $[[x]]$
$\to$ round down to nearest integer

Evaluate:

a) $[[1]]=1$ b) $[[-4]]=-4$ c) $[[1.3]]=1$ d) $[[-3.4]]=-4$



$\left.\begin{array}{l}\lim_{x\to 0^-}[[x]]=-1 \\ \lim_{x\to 0^+}[[x]]=0\end{array}\right] \neq$

$\lim_{x\to 0}[[x]]=DNE,$ $\lim_{x\to 0^-}[[x]] \neq \lim_{x\to 0^+}[[x]]$

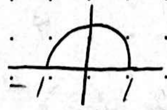
Existance of a limit:

$\lim_{x\to c} f(x)=L$ only if $\lim_{x\to c^-} f(x)=L$ and $\lim_{x\to c^+} f(x)=L$

Continuity on closed interval $[a,b]$

when f is cont. on (a,b) and $\lim_{x\to a^+} f(x)=f(a)$

and $\lim_{x\to b^-} f(x)=f(b)$ $\to \underbrace{\text{from right at } a \text{ \& at } b \text{ from left}}_{\text{continuous}}$

Ex 4: $f(x)=\sqrt{1-x^2}$  $[-1,1]$ $\lim_{x\to 1^-} f(x)=0$ continuous at $(-1,1)$

$\lim_{x\to -1^+} f(x)=0$

Determine if $f(x)$ is cont.:

Ex 6: a) $f(x)=x+\sin x$

$f(x)$ is everywhere cont.

c) $f(x)=\dfrac{x^2+1}{\cos x}$

$\to$ cont. @ $(-\infty,\infty)$

exept @ $\cos x=0$

and $x=\frac{\pi}{2}+\pi n$

b) $f(x)=3\tan x$ $x \neq \frac{\pi}{2}+n\pi$

$f(x)$ is continuous on

$\ldots\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)\ldots$

$\to f(x)$ is cont. on $(-\infty,\infty)$

exept at $x=\frac{\pi}{2}+n\pi$

Ex 7: Determine if cont.

a) $f(x)=\tan x$ b.

$(-\infty,\infty)$, except @ $x=\frac{\pi}{2}+\pi n$

1.4 HW p.79 3, 6, 9, 12, 14, 15, 17, 23, 25, 27, 28, 30, 43, 47, 51, 53,

61, 95

1.5 Double check 1.4 HW

Warm-up: If continuous. If not, find x-axis location for discont:

1. $f(x)=\begin{cases}\frac{x}{2}+\frac{5}{2}, & x\le 0\\ 2x+1, & x>0\end{cases}$ jump

Non removable discontinuity @ $x=0$, everywhere else continuous.

2. $f(x)=\frac{-x^2+2x+3}{x+3} \rightarrow \frac{(\cancel{x+3})(-x-1)}{\cancel{x+3}} \rightarrow \frac{-x-1}{1} \rightarrow x\neq -3$

Removable discontinuity @ $x=-3$, everywhere else continuous.

Intermediate Value Theorem Practice

(96) $f(x)=x^2-6x+8$, $[0,3]$, $f(c)=0$

$0=x^2-6x+8 \rightarrow (x-4)(x-2)=0 \rightarrow x=2,4$ $[0,3]$

$\boxed{c=2}$ 4 is not $[0,3]$

(97) $f(x)=x^3-x^2+x-2$, $[0,3]$, $f(c)=4$

$0=x^3-x^2+x-6$ Use $\frac{p}{q}$

p = factor of constant
q = factor of leading coefficient

$p=\pm1, \pm2, \pm3$
$q=\pm1$

$3\rfloor$ 1 −1 1 −6 / 3 6 7 / 1 2 7 | 1

$1\rfloor$ 1 −1 1 −6 / 1 0 1 / 1 0 1 | 1

$2\rfloor$ 1 −1 1 −6 / 2 2 6 / 1 1 3 | 10

C=2 because 2 is $[0,3]$

1.5 Limits at Infinity

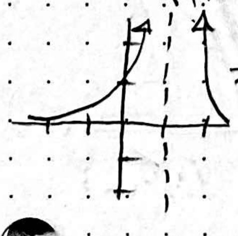
Sep. 3. 2024

HW. p88 Q: 15, 17, 23, 27, 39, 43, 55, 56, 65-68

Ex 1: Determine lim of function $x\to1$ from left & right

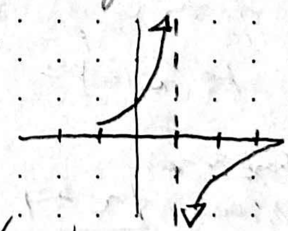
a) $\lim_{x\to1^+}\left(\frac{1}{(x-1)^2}\right)=\infty$

$\lim_{x\to1^-}\left(\frac{1}{(x-1)^2}\right)=\infty$



b) $\lim_{x\to1^+}f(x)=-\infty$

$\lim_{x\to1^-}f(x)=\infty$



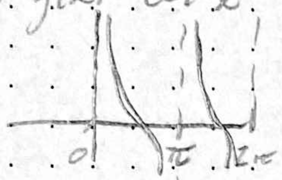
Verticle line $f(x)$ approaches but never touches
-Pre-Calc Verticle Asymptote

Calc: If $f(x)$ approaches infinity (or negative ") as x approaches c from left or right then $x=c$ is a verticle asymptote

Ex2:

(a) V.A. is @ $x=-1$ (b) $\frac{x^2+1}{x^2-1} \to$ V.A is @ $x=-1,1$

(c) $f(x)=\cot x$ V.A. is @ $x=\pi n$



Ex3: $f(x) = \frac{x^2+2x-8}{x^2-4} \to \frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} \to$ V.A. is @ $x=-2$

Ex4: $\frac{x^2-3x}{x-1} \to \frac{x(x-3)}{x-1}$ $x=1$ is V.A.

x	2	1.5
y	-2	-4.5

$\lim_{x\to 1^+} f(x) = -\infty$ $\lim_{x\to 1^-} f(x) = \infty$

Ex5: (a) $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right) \to 0+\infty = \infty$

(b) $\lim_{x\to 1} \frac{x^2+1}{\cot \pi n} = \frac{2}{-\infty} \to 0$ $\lim_{x\to 1} x^2+1 \to 2$ $\lim_{x\to 1} \cot \pi n \to -\infty$

(c) $\lim_{x\to 0^+} 3\cot x = \infty$

HW Questions 1.4

(14) $\lim_{x\to 10^+} \frac{|x-10|}{x-10} \to$ from right: $\frac{x-10}{x-10} = 1$ from left $\frac{-(x-10)}{x-10} = -1$

$\to 1$ because $x\to 10^+ \to$ from right

(15) $\lim_{\Delta x\to 0^+} \frac{\left(\frac{x}{x}\right)\frac{1}{x+\Delta x} - \frac{1}{x}\left(\frac{x+\Delta x}{x+\Delta x}\right)}{\Delta x} \to \lim_{\Delta x\to 0^+} \frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x} \to \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$

$\to \frac{-\Delta x}{x(x+\Delta x)(\Delta x)} \to \frac{-1}{x(x+\Delta x)} \to \frac{-1}{x(x+0)} \to \frac{-1}{x^2}$

Hypothesis: if you have a fraction, with abs$(x+a)$ look @ if $x\to b$ from left or right. If from right, abs$(x+a) = (x+a)$. if left, abs$(x+a) = -(x+a)$

Ex:

$\lim_{x\to 5^+} \frac{|x-5|}{(x-5)} \to \lim_{x\to 5^+} \frac{x-5}{x-5} = 1$

$\to \lim_{x\to 5^-} \frac{-(x-5)}{x-5} = -1$

if $\frac{\text{trig. func}(ax)}{ax} = a$

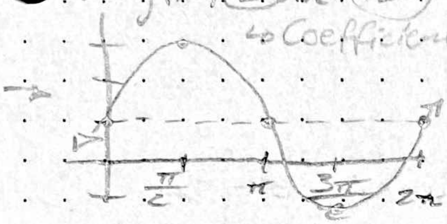
Ex: $\frac{\sin 3x}{3x} = 3$

Chapter P Review

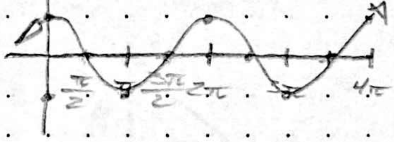
Solving Trig Equations

Trig Graphing:

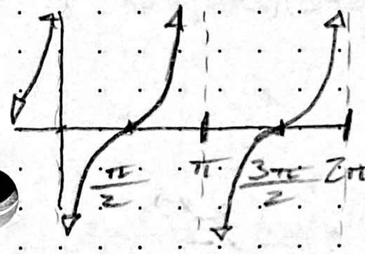
1. $f(x) = 2\sin x + 1$ Seperate from x, k value
↳ Coefficient of trig func is amplitude scalar



2. $f(x) = -\cos(x - \pi)$ negative coefficient → x-axis flip
↳ constant affecting x before trig. func.
↔ h value



3. $f(x) = \tan(x + \frac{\pi}{2})$



Solving Trig Equations:

(1) a) $-2\tan\theta\cos\theta + 2\tan\theta = 3\tan\theta$

↳ $-2\tan\theta\cos\theta - \tan\theta = 0$

↳ $\tan\theta(-2\cos\theta - 1) = 0$

$\tan\theta = 0$ when $\theta = \pi n, \in \mathbb{Z}$

$-2\cos\theta = 1 \rightarrow \cos\theta = -1/2$

$\rightarrow \cos\theta = -\frac{1}{2}$ when

$\theta = \frac{2\pi}{3} + 2\pi n$ & $\theta = \frac{4\pi}{3} + 2\pi n, n \in \mathbb{Z}$

b) $2\cot\theta + 1 = \cot^2\theta + 2$

↳ $\cot^2\theta - 2\cot\theta + (2-1) = 0$

↳ $\cot^2\theta - 2\cot\theta + 1 = 0$

$\rightarrow x = \cot\theta \rightarrow x^2 - 2x + 1 = 0$

$\rightarrow (x-1)(x-1) = 0$ or $(x-1)^2 = 0$

$x - 1 = 0 \rightarrow x = 1 \rightarrow \cot\theta = 1$

$\cot\theta = 1$ when $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

c) $6 - \sin\theta = 2\sin^2\theta + 5$

$\rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$

↳ $2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$

$\rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$

$2\sin - 1 = 0$ & $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ & $\sin\theta = -1$

when

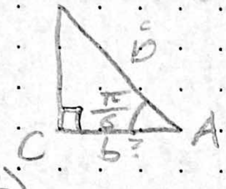
$\theta = \frac{\pi}{6} + 2\pi n$

$\frac{5\pi}{6} + 2\pi n$

$\theta = \frac{3\pi}{2} + 2\pi n, n \in \mathbb{Z}$

Right Triangle Trig

a). C is $90°$ find b if $A = \frac{\pi}{6}$ & $c = 10$



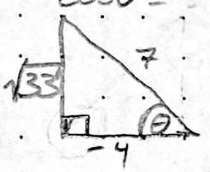
$\cos\frac{\pi}{6} = \frac{b}{10} \rightarrow \frac{\cos\frac{\pi}{6}}{10} = b \rightarrow \frac{\sqrt{3}}{2}(10) = b$

$\rightarrow 5\sqrt{3} = b$

b)

if $\frac{\pi}{2} < \theta < \pi$ & $\sec\theta = \frac{-7}{4}$, what is $\tan\theta$ & $\sin 2\theta$

$\cos\theta = -\frac{4}{7}\frac{\text{opp}}{\text{hyp}}$



$16 + b^2 = 49$

$b^2 = 33$

$\sin\theta = \frac{\sqrt{33}}{7}$

$\tan\theta = \frac{\sqrt{33}}{7} \cdot -\frac{7}{4}$

$\rightarrow -\frac{\sqrt{33}}{4}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\rightarrow 2\left(\frac{\sqrt{33}}{7}\right)\left(-\frac{4}{7}\right)$

$\rightarrow -\frac{8\sqrt{33}}{49}$

Points of Intersection

$x + y = 1 \rightarrow y = -x + 1$ $y = -0 + 1 \rightarrow 1$ $y = -3 + 1 \rightarrow -2$

$y = -x^2 + 2x + 1 \rightarrow -x + 1 = -x^2 + 2x + 1 \rightarrow x^2 - 3x + 0 \rightarrow (x-3)(x+0) = 0$

$x = 0, 3 \rightarrow (0,1), (3,-2)$

Transformation

$f(x) = \sqrt{x} \rightarrow f(x) = \underbrace{-2}_{x\text{-flip amp}}\underbrace{\sqrt{x+3}}_{\text{h}} + \frac{3}{2} \rightarrow k$

• Vert. streched • flipped on x-axis • left 3 • up $3/2$

Even-Odd funcs.

if $f(x) = -x^2 + 8x - 16$ is even or odd, explain:

Neither, not y or origin symmetry.

if $f(x) = \sqrt[3]{x}$ is even or odd, explain:

Odd, origin-sym. : $f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x}$

Symmetry How do you determine the symmetry of a graph of a function?

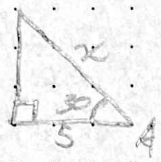
$f(-x) = f(x) \rightarrow$ even - y-axis sym.

$f(-x) = -f(x) \rightarrow$ odd - origin-sym.

if the function is consistent regardless of if $[x(-1)]$ or $(-1)[x]$, the function is odd - origin sym.

Otherwise, it's even - y-axis sym.

$A=30°$ $a=5$ find hyp

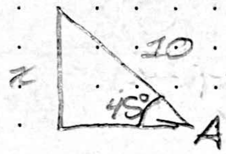


$\cos 30 = \frac{5}{x} \to \frac{\sqrt{3}}{2} = \frac{5}{x}$

$\to \frac{x}{1} \cdot \frac{\sqrt{3}}{2} = 5 \to \frac{x\sqrt{3}}{2} = 5 \to x\sqrt{3} = 10$

$\to x = \frac{10}{\sqrt{3}} \to \frac{10\sqrt{3}}{3}$

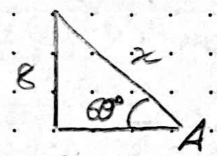
$A=45°$ hyp $=10$, find opp



$\sin 45° = \frac{x}{10} \to \frac{\sqrt{2}}{2} = \frac{x}{10} \to \frac{10}{1} \cdot \frac{\sqrt{2}}{2} \to$

$x = 5\sqrt{2}$

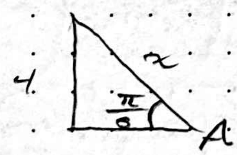
$A=60°$ opp $=8$ find hyp



$\sin 60 = \frac{8}{x} \to x \sin 60 = 8 \quad \frac{x}{1}\left(\frac{\sqrt{3}}{2}\right) \to \frac{x\sqrt{3}}{2} = 8$

$\to x\sqrt{3} = 16 \to x = \frac{16}{\sqrt{3}} \to \frac{16\sqrt{3}}{3}$

$A = \frac{\pi}{6}$ opp $=4$ find hyp



$\sin\frac{\pi}{6} = \frac{4}{x} \to \frac{1}{2} = \frac{4}{x} \to \frac{x}{2} = 4 \to x = 8$

$2\sin\theta - 1 = 0 \to \sin\theta = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi n$, $\theta = \frac{5\pi}{6} + 2\pi n$, $n \in \mathbb{Z}$

interval $[0, 2\pi)$: $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$2\cos^2\theta - 3\sin\theta = 0 \to 2(1-\sin^2\theta) - 3\sin\theta = 0$

$\to 2 - 2\sin^2\theta - 3\sin\theta = 0 \to -2\sin^2\theta - 3\sin\theta + 2 = 0$

$\to 2\sin^2\theta + 3\sin\theta - 2 = 0 \to (2\sin\theta - 1)(\sin\theta + 2) = 0$ $\quad -4$

$\sin\theta = \frac{1}{2}, -2$ when $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, General: $\left.\begin{aligned}\theta &= \frac{\pi}{6} + 2\pi n \\ &\frac{5\pi}{6} + 2\pi n\end{aligned}\right\} n \in \mathbb{Z}$

$\to$ no solution

$\sin(2\theta) + \sqrt{3}\cos(\theta) = 0 \to 2(\sin\theta)(\cos\theta) + \sqrt{3}\cos\theta = 0$

$\to \cos\theta(2\sin\theta + \sqrt{3}) = 0$

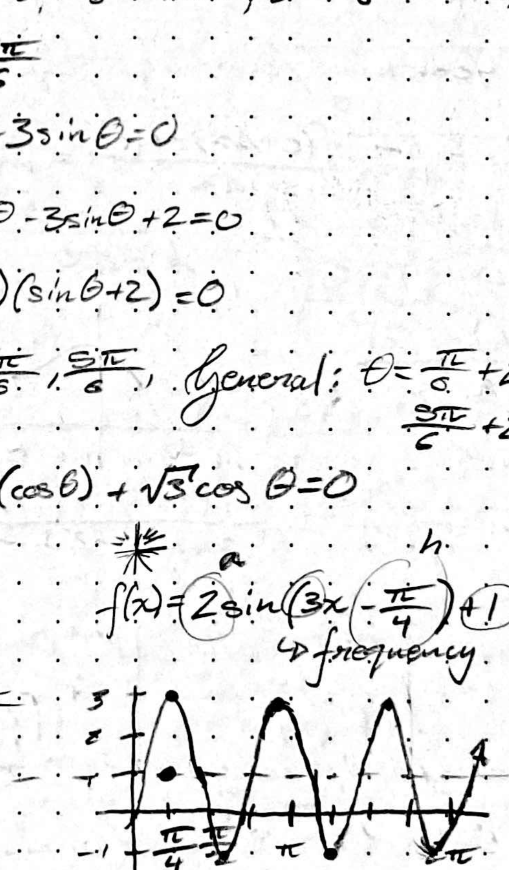
$\cos\theta = 0 \qquad \sin\theta = -\frac{\sqrt{3}}{2}$

when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ & when $\theta = \frac{5\pi}{3}, \frac{4\pi}{3}$

a h

$f(x) = 2\sin\left(3x - \frac{\pi}{4}\right) + 1$ k

$\to$ frequency



2.1 Warm-Up   Find the limit   Write $\left\{\lim_{x\to 0} \dots\right\}$ until you evaluate limit!

$$\lim_{h\to 0}\left\{\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\right\} \to \left(\frac{x}{x}\right)\left(\frac{3}{x+h}\right) - \frac{3}{x}\left(\frac{x+h}{x+h}\right)$$

$$\to \frac{3x-3x-3h}{x(x+h)} \to \frac{\frac{-3h}{x(x+h)}}{h}$$

$$\to \frac{-3h}{x(x+h)}\cdot\frac{1}{h} \to \frac{-3}{x(x+h)} \to \frac{-3}{x^2} = \lim_{h\to 0}\left\{\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\right\}$$

$$\lim_{h\to 0}\frac{\frac{-3h}{x(x+h)}}{h} \to \lim_{h\to 0}\frac{-3h}{x(x+h)}\cdot\frac{1}{h} \to \lim_{h\to 0}\frac{-3}{x(x+h)} = \frac{-3}{x^2}$$

2.1 The Derivative & Tangent Lines

Short Cut to find $f'(x)$ or derivative or slope or rate of change

$f(x)=ax^n \to f'(x)=ax^{n-1}$

ex: 1 $f(x)=x^2+4x$

$f'(x)=2x^1+4x^0 \to 2x+4$ $f'(1)=2(1)+4 \to 6$ → slope @ $x=1$

What is a tangent?

Straight line that crosses the graph @ 1 point

Secant line?

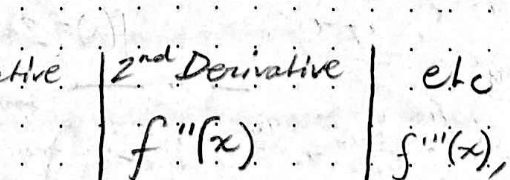
Line touching/crossing graph 2 times

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(c+\Delta x)-f(c)}{\Delta x} = m \qquad m = \text{slope}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Difference Quotient $h=\Delta x$

$$\frac{f(x+h)-f(x)}{h}$$



Slope form.

$\to \frac{y_2-y_1}{x_2-x_1}$ $m_{sec} = \frac{f(x+h)-f(x)}{x+h-x} \to \frac{f(x+h)-f(x)}{h} \to \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

makes sec line into tangent

Symbols

Function	1st Derivative	2nd Derivative	etc
$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$
y	y' or $\frac{dy}{dx}$	y'' or $\frac{d^2x}{d^2x}$	

p. 103: 1-4, 7, 9, 17, 21, 23, 29, 33, 37, 39-45

Find slope of tan: 1 find derivative, 2 plug in x

Ex 1: $f(x)=2x-3$ when $c=2$

$y=mx+b \to m=2$

Ex 2: $f(x)=x^2+1$ @ $(0,1)$ & $(-1,2)$ [0 & -1]

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \to \lim_{h\to 0}\frac{(x+h)^2+1-(x^2+1)}{h}$$

$$\to \frac{\cancel{x^2}+2xh+h^2\cancel{+1}-\cancel{x^2}\cancel{-1}}{h} \to \frac{2xh+h^2}{h} \to \frac{\cancel{h}(2x+h)}{\cancel{h}} \to 2x+h$$

$$\lim_{h\to 0} \to 2x+0 \to 2x$$

$f'(x)=2x$ Plug in x:

At $(0,1)$, $m=f'(0)=2(0)=0$

At $(-1,2)$, $m=f'(-1)=2(-1)=-2$

Ex 3: Use def. of deriv. (long way)

$$f(x)=x^2-2x+1 \to \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \to \lim_{h\to 0}\frac{(x+h)^2-2(x+h)+1-(x^2-2x+1)}{h}$$

$$\to \lim_{h\to 0}\frac{\cancel{x^2}+2hx+h^2\cancel{-2x}-2h\cancel{+1}-\cancel{x^2}+\cancel{2x}\cancel{-1}}{h} \to \frac{2hx+h^2-2h}{h}$$

$$\to \lim_{h\to 0}\frac{h(2x+h-2)}{h} \to \lim_{h\to 0} 2x+h-2 \to 2x+0-2$$

derivative: $2x-2$ $(1)x^2-2x+1$

Short cut check: $2x-2 \leftarrow 2(1)x-2^0$

b) $f(x)=x^3+2x$ $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^3+2(x+h)-(x^3+2x)}{h}$$

$(x+h)^3$

$1x^3h^0$, $3x^2h^1$, $3x^1h^2$, $1x^0h^3$

1; 1 2 1; 1 3 3 1

$1x^3$, $3x^2h$, $3xh^2$, $1h^3$

$(x+h)^3 = 1x^3+3x^2h+3xh^2+h^3$

$$\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{+2x}+2h-\cancel{x^3}-\cancel{2x}}{h} \to \lim_{h\to 0}\frac{h(3x^2+3xh+h^2+2)}{h}$$

$$\to \lim_{h\to 0} 3x^2+3xh+h^2+2 \to 3x^2+3x(0)+0^2+2 \to \boxed{3x^2+2=f'(x)}$$

Ex. 4: $f(x)=\sqrt{x}$ @ $(1,1)$ & $(4,2)$ discuss f at $(0,0)$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h},\ \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$\to \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \to \lim_{h\to 0}\frac{h}{h(\ \ )}\ \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \to \frac{1}{\sqrt{x+0}+\sqrt{x}} \to \frac{1}{2\sqrt{x}}$$

m at $(1,1) \to f'(1)=\frac{1}{2\sqrt{1}} \to \frac{1}{2}$ m @ $(4,2) \to f'(4)=\frac{1}{2\sqrt{4}} \to \frac{1}{4}$

2.1 part 2 Sept. 12. 2024

Ex 5: Deriv. with ~~respect~~ to "t" for $y=\frac{2}{t}$

$\rightarrow \frac{dy}{dt} = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \rightarrow \lim_{h\to 0} \frac{\frac{(t)\,2}{(t)\,t+h} - \frac{2}{t}\left(\frac{t+h}{t+h}\right)}{h}$

$\rightarrow \lim_{h\to 0} \frac{\frac{2t}{t(t+h)} - \frac{2t-2h}{t(t+h)}}{h} \rightarrow \lim_{h\to 0} \frac{\frac{\cancel{2t}-\cancel{2t}-2h}{t(t+h)}}{\frac{h}{1}}$

$\rightarrow \lim_{h\to 0} \frac{-2h}{t(t+h)} \cdot \frac{1}{h} \rightarrow \lim_{h\to 0} \frac{-2}{t(t+h)} \rightarrow \frac{-2}{t^2} = \frac{dy}{dt}$

Quick Check: $y=\frac{2}{t} = 2t^{-1} \rightarrow y' = -2t^{-2} = \frac{-2}{t^2}$

to 1 less

Ex 6:

Find an equation which is tangent to $f(x) = 2x^2 - 3x$ @ $x=-1$

How to find a tangent line:

1) find $f'(x)$: $f'(x) = \lim_{h\to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2-3x)}{h}$

$\rightarrow \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-3x} - 3h \cancel{-2x^2} \cancel{+3x}}{h}$

$\rightarrow \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} \rightarrow \lim_{h\to 0} \frac{h(4x + 2h - 3)}{h}$

$\rightarrow \lim_{h\to 0} 4x + 2h - 3 \rightarrow 4x + 2(0) - 3 \rightarrow 4x - 3 = f'(x)$

2) find m @ $x=-1$ $\quad f'(-1) = 4(-1) - 3 = -7 = m$ or $\frac{-7}{1}$

3) find y_1 $\quad 2(-1)^2 - 3(-1) \rightarrow 2+3 \rightarrow 5 = y_1$

4) Write equation in the form $y - y_1 = m(x - x_1)$

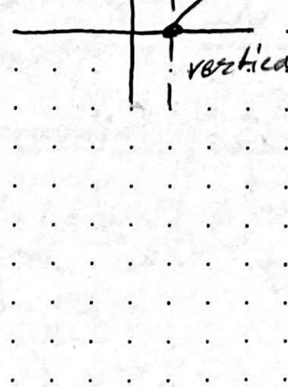
Slope form $\rightarrow y - 5 = -\frac{7}{1}(x+1)$ or $y = -7(x+1) + 5$

sharp tip in graph = cusp $\rightarrow$ cannot be differentiated

$\rightarrow$ no defined slope

$\rightarrow$ uses vertical tangent line

Ex:



vertical tangent line, undefined slope

2.2 Warm-up

a) $g(x) = -3x^2 + x - 2 \rightarrow g'(x) = \lim_{h\to 0} \frac{-3(x+h)^2 + (x+h) - 2 - (-3x^2+x-2)}{h}$

$\rightarrow \lim_{h\to 0} \frac{-3x^2 - 6xh - 3h^2 + x + h - 2 + 3x^2 - x + 2}{h} \rightarrow \lim_{h\to 0} \frac{-6xh - 3h^2 + h}{h}$

$\rightarrow \lim_{h\to 0} \frac{h(-6x - 3h + 1)}{h} \rightarrow \lim_{h\to 0} -6x - 3h + 1 \rightarrow -6x + 1 = g'(x) \rightarrow$ plug in $x = 1$ ↳ gives slope

b) $-3(1)^2 + 1 - 2 \rightarrow -4 \rightarrow (1, -4)$ $y + 4 = -5(x-1)$ $y = -5(x-1) - 4$

2.2 Basic Differentiation Rules & Rates of change

The derivative of a constant function is $\underline{0}$
↳ as long as it's real

Ex 1

a) $y = 7 \rightarrow y' = 0$ b) $y = 0 \rightarrow y' = 0$

c) $y = -3 \rightarrow y' = 0$ d) $y = k\pi^2$, k is constant $\rightarrow y'(x) = 0$

Power rule: If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx} x^n = nx^{n-1}$$

Ex 2

a) $y = x^3 \rightarrow y'(x) = 3x^2$ b) $g(x) = \sqrt[3]{x} \rightarrow x^{\frac{1}{3}} \rightarrow g'(x) = \frac{1}{3} x^{-2/3}$

c) $y = \frac{1}{x^2} \rightarrow x^{-2} \rightarrow y'(x) = -2x^{-3}$ or $\boxed{\frac{-2}{x^3}}$

Ex 3

$f(x) = x^2$ when $x = -2$ $(-2, 4)$ $f'(x) = 2x \rightarrow m = -4 \rightarrow$ tangent:
$y - 4 = -4(x+2)$

Constant Multiple Rule

If f is a differentiable function & c is real, then cf is also and $\frac{d}{dx}[cf(x)] = cf'(x)$

$$\rightarrow \boxed{\frac{d}{dx}[cx^n] = cnx^{n-1}}$$

Ex 4:

a) $y = 5x^3 \rightarrow \boxed{y' = 15x^2}$ b) $y = \frac{2}{x} \rightarrow 2x^{-1} \rightarrow y' = -2x^{-2}$ or $\boxed{\frac{-2}{x^2}}$

c) $f(t) = \frac{4t^2}{5} \rightarrow f'(x) = \frac{4}{5} \cdot 2t \rightarrow \boxed{\frac{8t}{5}}$

d) $y = 2\sqrt{x} \rightarrow 2x^{\frac{1}{2}} \rightarrow y' = 1x^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{x}}$

(Pick 12)

HW: p.114: 1, 4, 13, 23, 28, 35, 39, 41, 43, 49, 54, 59, ~~61~~ 60, 66, 67, 70, 79

e) $y = \frac{1}{2\sqrt[3]{x^2}} \to \frac{1}{2x^{\frac{2}{3}}} \to \frac{x^{-2/3}}{2}$ $f'(x) = \frac{1}{2}\cdot\left(-\frac{2}{3}\right)x^{-5/3}$

$\to -\frac{1}{3}x^{-5/3} \to -\frac{x^{-5/3}}{3} \to -\frac{1}{3\sqrt[3]{x^5}}$

f) $y = -\frac{5x}{2} \to -\frac{5}{2}\cdot 1x$

Ex 5:

a) $y = \frac{5}{2x^3} \to \frac{5}{2}\cdot\frac{x^{-3}}{1} \to \frac{5x^{-3}}{2}$ $\frac{dy}{dx} = \frac{5}{2}\cdot\frac{-3x^{-4}}{1}$

$\to \frac{5}{2}\cdot\frac{-3x^{-4}}{1} \to \frac{-15x^{-4}}{2} \to -\frac{15}{2x^4}$

b) $y = \frac{5}{(2x)^3} \to \frac{5}{8x^3} \to \frac{5}{8}\cdot\frac{x^{-3}}{1} \to \frac{5}{8}\cdot\frac{-3x^{-4}}{1} \to \frac{-15x^{-4}}{8}$

$\to \frac{-15}{8x^4}$

c) $y = \frac{7}{3x^{-2}} \to \frac{7}{3}\cdot\frac{x^2}{1} \to \frac{dy}{dx} = \frac{7}{3}\cdot\frac{2x}{1} \to \frac{14x}{3} \to \frac{14}{3x^{-1}}$

d) $y = \frac{7}{(3x)^{-2}} \to 7(3x)^2 \to 7(9x^2) \to 63x^2 \to \frac{d}{dx}$

Position Function, Velocity & Acceleration

- $s(t) = -16t^2 + v_0 t + s_0$
-
-

Sum & Difference Rules

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

Ex 6:

a) $f(x) = x^3 - 4x + 5$ $f'(x) = 3x^2 - \cancel{4x}\ 1 + 0$ $3 - \frac{1}{x^2}$

$\to f'(x) = 3x^2 - 1$

b) $g(x) - \frac{x^4}{2} + 3x^3 - 2x$ $g'(x) = -2x^3 + 9x^2 - 2$

c) $y = \frac{3x^2 - x + 1}{x} \to 3x - 1 + \frac{1}{x}$ $y' = 3\cancel{x}^0 - 0 - x^{-2} \to 3 - x^{-2}$

Derivatives of Sine & Cosine

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Ex 7:

a) $y = 2\sin x \to y' = 2\cos x$ b) $y = \frac{\sin x}{2}$ $y' = \frac{\cos x}{2}$

c) $y = x + \cos x \to y' = 1\cancel{x}^0 + (-\sin x) \to y' = 1 - \sin x$

d) $\cos x - \frac{\pi}{3}\sin x$ $\frac{d}{dx} = -\sin x - \frac{\pi}{3}\cos x$

2.3 Product & Quotient Rules Sept. 19. 2024

Warm-up

1) $f(x) = 2\sqrt{x} - 4x^5 \rightarrow 2x^{1/2} - 4x^5 \rightarrow 1x^{-1/2} - 20x^4$

$\rightarrow f'(x) = \frac{1}{\sqrt{x}} - 20x^4$

2) $y = \frac{3}{(3x)^3} \rightarrow \frac{3x^{-3}}{27} \rightarrow y' = \frac{3}{27} \cdot \frac{-3x^{-4}}{1} \rightarrow \frac{-1}{3x^4}$ or $\frac{-x^{-4}}{3}$

Position, Velocity, & Acceleration

Position
Units: ft

$S(t) = -16t^2 + V_0 t + S_0$

initial velocity — initial height

Velocity
Units: $\frac{ft}{sec}$

$S'(t) = v(t) = -32t + V_0$

Acceleration
Units: $\frac{\left(\frac{ft}{sec}\right)}{sec}$ or ft/sec^2

$S''(t) = v'(t) = a(t) = -32$

The product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Ex 1: Derive

$h(x) = \underbrace{(3x - 2x^2)}_{f(x)}\underbrace{(5+4x)}_{g(x)} \rightarrow h'(x) = (3x - 2x^2)(4) + (5+4x)(3-4x)$

$\rightarrow h'(x) = 12x - \underline{8x^2} + 15 - 20x + 12x - \underline{16x^2} \rightarrow h'(x) = -24x^2 + 4x + 15$

Ex 2: Derive

$y = 3x^2 \sin x \rightarrow y' = (3x^2)(\cos x) + (6x)(\sin x)$

Ex 3: Derive

$y = 2x\cos x - 2\sin x \rightarrow y' = (2x)(-\sin x) + 2\cos x - 2\cos x \quad y' = -2x\sin x$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

(high: $f(x)$, low: $g(x)$)

$\frac{\text{low } d'\text{high} - \text{high } d'\text{low}}{\text{low}^2}$

Ex 4: Derive

$y=\frac{5x-2}{x^2+1} \to y'=\frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)(x^2+1)} \to \frac{5x^2+5-10x^2+4x}{(x^2+1)^2}$ ← Don't foil!

$\to y'=\frac{-5x^2+4x+5}{(x^2+1)^2}$

Ex 5: Find tan. line

$f(x)=\frac{3-\left(\frac{1}{x}\right)}{x+5}$ @ $(-1,1)$ $\to \frac{3-x^{-1}}{x+5} \to f'(x)=\frac{(x+5)(1x^{-2})-(3-x^{-1})(1)}{(x+5)^2}$

$\to f'(x)=\frac{\frac{x+5}{x^2}-\left(3-\frac{1}{x}\right)}{(x+5)^2}$

m @ $-1 \to f'(-1)=\frac{\frac{4}{1}-(4)}{(-1+5)^2} \to \frac{0}{16} \to m=0$

tangent line equation: $y-1=0(x+1) \to \boxed{y=1}$ ✓

Ex 6: Derive

a) $y=\frac{x^2+3x}{6} \to \frac{1}{6}\cdot x^2+3x \to \frac{1}{6}\cdot 2x+3 \to \frac{2x+3}{6}=y'$

b) $y=\frac{5x^4}{8} \to \frac{5}{8}\cdot\frac{x^4}{1} \to \frac{5}{8}\cdot\frac{4x^3}{1} \to \frac{20x^3}{8} \to \frac{5x^3}{2}=y'$

c) $y=\frac{-3(3x-2x^2)}{7} \to -3\left(\frac{3x}{7x}-\frac{2x^2}{7x}\right)=-3\left(\frac{3}{7}-\frac{2x}{7}\right)$

d) $y=\frac{9x^{-2}}{5} \to y'=\frac{-18x^{-3}}{5} \to \frac{-18}{5x^3}$

Derivative of tan x? $\to \frac{\sin x}{\cos x} \to y'=\frac{\cos x(\cos x)-\sin x(-\sin x)}{(\cos x)^2}$

$\to \frac{\cos^2 x+\sin^2 x}{(\cos x)^2} \to \frac{1}{\cos^2 x}=\sec^2 x$

Derivative of cot x? $\to \frac{\cos x}{\sin x} \to -\csc^2 x$ $\quad y'=\frac{\sin x(-\sin x)-\cos x(\cos x)}{\sin^2 x}$

$\to \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} \to \frac{-1(\sin^2 x+\cos^2 x)}{\sin^2 x} \to \frac{-1}{\sin^2 x} \to -\csc^2 x$

Derivative of Trig. functions

$\frac{d}{dx}\sec x=\sec x\tan x$ $\qquad \frac{d}{dx}\csc x=-\csc x\cot x$

$\frac{d}{dx}\cos x=-\sin x$ $\qquad \frac{d}{dx}\sin x=\cos x$

$\frac{d}{dx}\tan x=\sec^2 x$ $\qquad \frac{d}{dx}\cot x=-\csc^2 x$

Ex 7: Derive

a) $y=x-\tan x \to y'=1-\sec^2 x=-\tan^2 x$

b) $y=x|\sec x \to y'=x(\sec x\tan x)+\sec x$

$y'=x\sec x\tan x+\sec x$

Ex. 8: Differentiate & Prove both sides

$$y = \frac{1-\cos x}{\sin x} = \csc x - \cot x$$

$$\frac{dy}{dx} = \frac{\sin x(\sin x) - (1-\cos x)(\cos x)}{\sin^2 x} = -\csc x \cot x + \csc^2 x$$

$$\frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x}$$

$$\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$$

$$\csc^2 x - \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$\csc^2 x - \cot x \csc x = -\csc x \cot x + \csc^2 x$$

HW: p. 125 Q: 1, 5, 11, 15, 19, 25, 30, 37, 41, 50, 60, 65a, 69, 79

2.4 Chain Rule

Warm up $f(x)=\sqrt{x+2}$ & $g(x)=x^2+4$ find $f\circ g$ & $g\circ f$

$\sqrt{(x^2+4)+2} = f\circ g$

$\rightarrow \sqrt{x^2+6}$

$(\sqrt{x+2})^2+4 = g\circ f$

$x+2+4 \rightarrow x+6$

$h(x)=\sqrt{x}$ $k(x)=x^2+6$

$h(k(x)) = \sqrt{x^2+6}$

Without the Chain Rule

$y=x^2+1$
$y=\sin x$
$y=3x+2$
$y=x+\tan x$

With the Chain Rule

$y=\sqrt{x^2+1}$
$y=\sin 6x$
$y=(3x+2)^5$
$y=x+\tan x^2 \neq \tan^2 x$

The Chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$$

Ex 1.

a) $y=\dfrac{1}{x+1}$ $u=x+1$ $y=\dfrac{1}{u}$

b) $y=\sin 2x$ $u=2x$ $y=\sin u$

c) $y=\sqrt{3x^2-x+1}$ $u=3x^2-x+1$ $y=\sqrt{u}$ or $u^{1/2}$

d) $y=\tan^2 x$ $u=\tan x$ $y=u^2$
$=(\tan x)^2$

Ex. 2

$y=(x^2+1)^3$ $u=x^2+1$ $y=u^3$

$\frac{dy}{dx}=3(x^2+1)^2(2x) = \boxed{6x(x^2+1)^2}$

General Power Rule

$$\frac{dy}{dx}=n[u(x)]^{n-1}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[u^n]=nu^{n-1}u'$$

Ex 3 $f(x)=(3x-2x^2)^3$ $u=3x-2x^2$ $y=u^3$

$f'=3(3x-2x^2)^2(3-4x) \rightarrow \boxed{9-12x(3x-2x^2)^2}$

Ex 4 $f(x)=\sqrt[3]{(x^2-1)^2}$ where $f'(x)=0$ & where $f'(x)$ dne

$f(x)=(x^2-1)^{2/3}$ $u=x^2-1$ $y=u^{2/3}$ $f'(x)=\frac{2}{3}(x^2-1)^{-1/3}(2x)$

$\rightarrow f'(x)=\frac{4}{3}x(x^2-1)^{-1/3}$ or $\dfrac{4x}{3\sqrt[3]{x^2-1}}$

$f'(x) \rightarrow 0 = \frac{4x}{3\sqrt[3]{x^2-1}} \rightarrow 0 = 4x \rightarrow 0 = x$

$(0, 1)$ original $f(x) \rightarrow$ eval.: $\sqrt[3]{(0^2-1)^2} = 1$ } Finding where $f'(x) = 0$

$\hookrightarrow$ where $f'(x)$ is 0

$f'(x)$ does not exist when the denominator is 0

$f'(x)$ denominator $\rightarrow 3\sqrt[3]{x^2-1} = 0 \rightarrow \sqrt[3]{x^2-1} = 0$

$\rightarrow x^2 - 1 = 0 \rightarrow x^2 = 1 \rightarrow x = \pm 1$

$(1, 0)$ $(-1, 0)$

Points where $f'(x)$ dne

$f(1) = \sqrt[3]{(1^2-1)^2} = 0$

$f(-1) = \sqrt[3]{(-1^2-1)^2} = 0$

HW: P. 136 Q: 1, 5, 10, 11, 13, 21, ~~27~~, ~~31~~, 45, 59, 67, ~~77~~, (81-89) odd

2.4 Day 2 Sept. 26 2024

Ex 5:

$g(t) = \frac{-7}{(2t-3)^2}$ $u = 2t - 3$ $y = \frac{-7}{u^2} \rightarrow y = -7u^{-2}$

$\frac{d}{dx} = 14(2t-3)^{-3}(2) \rightarrow \boxed{28(2t-3)^{-3}} \rightarrow \frac{28}{(2t-3)^3}$

Ex. 6: Product & Chain Rule

$f(x) = x^2\sqrt{1-x^2} \rightarrow x^2(1-x^2)^{1/2} \rightarrow u = 1 - x^2$ $y = u^{1/2}$

$f'(x) = x^2\left(\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right) + 2x(1-x^2)^{1/2}$

$= \frac{-x^3}{\sqrt{1-x^2}} + 2x\sqrt{1-x^2}$

Ex 7:

$f(x) = \frac{x}{\sqrt[3]{x^2+4}} \rightarrow x(x^2+4)^{-1/3} \rightarrow u = x^2 + 4$ $y = u^{-1/3}$

$f'(x) = x\left(-\frac{1}{3}(x^2+4)^{-4/3}(2x)\right) + 1\left((x^2+4)^{-1/3}\right) = -\frac{2x^2}{3(x^2+4)^{4/3}} + \frac{1}{(x^2+4)^{1/3}}$

Ex 8:

$y = \left(\frac{3x-1}{x^2+3}\right)^2$ $y' = 2\left(\frac{3x-1}{x^2+3}\right)^1\left(\frac{(x^2+3)(3) - (3x-1)(2x)}{(x^2+3)^2}\right)$

$= \frac{(6x-2)(3x^2+9-6x^2+2x)}{(x^2+3)^3}$

Trig fncs with chain rule

$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$

$\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

$\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

Ex 9

a) $y = \sin 2x \to y' = (\cos(2x))(2) \to 2\cos 2x$

b) $y = \cos(x-1)$ $y' = -\sin(x-1)(1) \to -\sin(x-1)$

c) $y = \tan 3x$ $y' = (\sec^2(3x))(3) \to 3\sec^2(3x)$

Ex. 11:

$f(t) = \sin^3 4t \to (\sin(4t))^3$ $u = \sin 4t$ $y = u^3$

$\to y' = 3(\sin 4t)^2 (\cos(4t)(4)) \to 3(\sin 4t)^2 4\cos(4t) \to 12\sin^2(4t)\cos(4t)$

Ex. 11.5:

$f(x) = \sec^4(3x) \to (\sec 3x)^4$ $u = \sec 3x$ $y = u^4$

$u' = \sec 3x \tan 3x(3) = 3(\sec 3x \tan 3x)$

$f'(x) = 4(\sec 3x)^3(3(\sec 3x \tan 3x)$

$\to 12\sec^3 3x(\sec 3x \tan 3x)$

HW: p 136 (1, 8, 10, 11, 13, 21, 27, 31, 45, 59, 67, 77, 81-89 odd)

2.5 Implicit Differentiation

Oct. 1. 2024

Warm up → 1) $f(x) = \sqrt[3]{2x-1}$ → $(2x-1)^{1/3}$ @ $x = -1$ find tangent:

$\to f'(x) = \frac{1}{3}(2x-1)^{-2/3}(2) \to \frac{7}{3}(2x-1)^{-2/3} \to \boxed{\frac{7}{3\sqrt[3]{(2x-1)^2}} = f'(x)}$

$\to \frac{2}{3}(2x-1)^{-2/3} \to \frac{2}{3(2x-1)^{2/3}} \to f'(-1) = \frac{2}{3(2(-1)-1)^{2/3}}$

$m = 0.32$ @ $x = -1$, $y = -1.442$

$\to y + 1.442 = 0.32(x+1)$

2) $g(x) = \sec 3x + \cot^2 x$ $\qquad g'(x) = 3\sec 3x \tan 3x - 2\cot x \csc^2 x$

2.5 Implicit Equation: Equation where you cannot solve for y

Functions in implicit form:

$xy^2 + y^3 = x + 4$ → can't solve for y.

Ex: Find what y equals

$x^2 + y^2 = 25$

Unnecessarily complicated

$\sqrt{y^2} = \sqrt{25 - x^2} \to y = \pm\sqrt{25 - x^2}$

When deriving implicitly, differentiation is taking place with respect to x

$\frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}(y) = \frac{dy}{dx}$

$\frac{d}{dx}(y^2) = 2y^1\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(y^8) = 8y^7\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(\sin y) = \cos y\left(\frac{dy}{dx}\right)$

$\frac{d}{dx}(z) = \frac{dz}{dx}$

$\frac{d}{dx}(z^3) = 3z^2\left(\frac{dz}{dx}\right)$

$\frac{d}{dx}(\sec z) = \sec z \tan z\left(\frac{dz}{dx}\right)$

Ex 1:

a) $\frac{d}{dx}(x^4) = 4x^3$ $\qquad$ b) $\frac{d}{dx}(y^4) = 4y^3\left(\frac{dy}{dx}\right)$

c) $\frac{d}{dx}(x+3y) = \left(1 + 3\left(\frac{dy}{dx}\right)\right)$ $\qquad$ d) $\frac{d}{dx}(xy^2) = x2y\frac{dy}{dx} + 1y^2$

$\to 2xy\frac{dy}{dx} + y^2$

Guidelines:

1) Differentiate both sides of the equation with respect to x

2) Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation & move all other terms to the right side of the equation

3) Factor $\frac{dy}{dx}$ out of the left side of the equation

4) Solve for $\frac{dy}{dx}$

Ex 2: Find $\frac{dy}{dx}$ given:

$y^3+y^2-5y-x^2=-4 \rightarrow \frac{d}{dx}(y^3+y^2-5y-x^2)=\frac{d}{dx}(-4)$

$\rightarrow 3y^2\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)-5\left(\frac{dy}{dx}\right)-2x=0$

$\rightarrow 3y^2\left(\frac{dy}{dx}\right)+2y\left(\frac{dy}{dx}\right)-5\left(\frac{dy}{dx}\right)=2x$

$\rightarrow \frac{dy}{dx}(3y^2+2y-5)=2x \rightarrow \frac{dy}{dx}=\left(\frac{2x}{3y^2+2y-5}\right)$

Ex 3:

a) $x^2+y^2=0 \rightarrow \sqrt{y^2}=\sqrt{-x^2} \rightarrow$ non-real
ignore for calc

b) $x^2+y^2=1 \rightarrow y=\pm\sqrt{1-x^2}$
differentiable

c) $x+y^2=1 \rightarrow y=\pm\sqrt{1-x}$
differentiable

Ex 4: determine slope @ $(\sqrt{2}, -1/\sqrt{2})$ of

$x^2+4y^2=4 \rightarrow \frac{d}{dx}(x^2+4y^2)=\frac{d}{dx}(4) \rightarrow 2x+8y\left(\frac{dy}{dx}\right)=0$

$\rightarrow \frac{dy}{dx}(8y)=-2x \rightarrow \frac{dy}{dx}=-\frac{2x}{8y} \rightarrow -\frac{x}{4y}$

$m@\left(\sqrt{2},-\frac{1}{\sqrt{2}}\right)=\frac{dy}{dx}\Big|_{\left(\sqrt{2},-\frac{1}{\sqrt{2}}\right)}=\frac{-\sqrt{2}}{-4/\sqrt{2}} \rightarrow \frac{-\sqrt{2}}{1}\cdot\frac{\sqrt{2}}{-4}=\frac{-2}{-4}$

$\hookrightarrow \frac{1}{2}=m$

Ex 5: Derive implicitly.

$xy^3+y^2+2x=-10 \rightarrow \frac{d}{dx}(xy^3+y^2+2x)=\frac{d}{dx}(-10)$

$\rightarrow \left(x\left(3y^2\left(\frac{dy}{dx}\right)\right)+1(y^3)\right)+2y\left(\frac{dy}{dx}\right)+2=0$

$3xy^2\left(\frac{dy}{dx}\right)+y^3+2y\left(\frac{dy}{dx}\right)+2=0 \rightarrow \frac{dy}{dx}(3xy^2+2y)=-y^3-2$

$\rightarrow \frac{dy}{dx}=\frac{-2-y^3}{(3xy^2+2y)}$

Ex 7: Second derivative implicitly ★ On quiz & test! ★

$x^2+y^2=25$ find $\frac{d^2y}{dx^2}$

$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25) \rightarrow 2x+2y\left(\frac{dy}{dx}\right)=0$

$\rightarrow \left(\frac{dy}{dx}\right)(2y)=-2x \rightarrow \frac{dy}{dx}=\frac{-2x}{2y} \rightarrow \frac{-x}{y}$ first derivative

$\frac{d^2y}{dx^2}=\frac{y(-1)-(-x)\left(\frac{dy}{dx}\right)}{y^2}$ $=\frac{\frac{y}{y}\cdot\frac{-y}{1}+x\left(\frac{-x}{y}\right)}{y^2}$

$\rightarrow \frac{\frac{-y^2-x^2}{y}}{y^2} \rightarrow \frac{\frac{(-1)y^2+x^2}{y}}{y^2} = \frac{\frac{(-1)25}{y}}{y^2}$ (check original equation) $\rightarrow \frac{-25}{y^3}$

HW: 145 (1, 5, 9, 11, 21, 27, 29, 34, 37, 60, 68)

2-6

WU: $x^3y^3 - y = x \rightarrow \left(x^3\left(3y^2\frac{dy}{dx}\right) + 3x^2\left(y^3\right)\right) - \frac{dy}{dx} = 1$

$\rightarrow \frac{dy}{dx}\left(x^3 3y^2 - 1\right) = 1 - 3x^2y^3$

$\rightarrow \boxed{\frac{dy}{dx} = \frac{1-3x^2y^3}{(x^3 3y^2 - 1)}}$

@ (0,0)

$\rightarrow \frac{1-3(0)^2(0)^3}{0^3 3(0)^2 - 1} \rightarrow \frac{1}{-1} \rightarrow -1$

$\boxed{\left.\frac{dy}{dx}\right|_{(0,0)} = -\frac{1}{1}}$

Differentiating with respect to "t"
using implicit differentiation

$\frac{d}{dt}(x) = \frac{dx}{dt} \quad \frac{d}{dt}(x^2) \rightarrow 2x\frac{dx}{dt} \quad \frac{d}{dt}(x^5) \rightarrow 5x^4\frac{dx}{dt}$

$\frac{d}{dt}(y) = \frac{dy}{dt} \quad \frac{d}{dt}(y^3) = 3y^2\frac{dy}{dt} \quad \frac{d}{dt}(z) = \frac{dz}{dt} \quad \frac{d}{dt}(V) = \frac{dV}{dt}$

$\frac{d}{dt}(r^2) = 2r\frac{dr}{dt}$

2.6 Related Rates

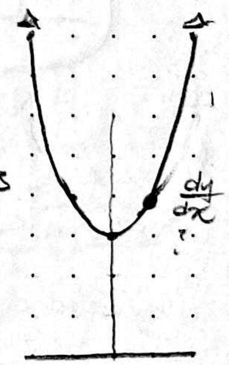
Related Rates - slopes of 2 or more related variables that are changing with respect to time

Ex: Pouring water

Constants: Size of containers (Radius), Color, amount of water existing

Variable (Things that change): Liquid in 2nd container, Air decreased in 2nd container, flow rate, "shape of water"

Related Rates: $\frac{dV}{dt}$ $\frac{dh}{dt}$



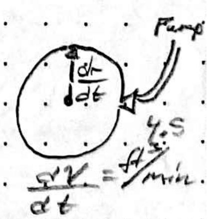
Ex 1: x & y are diff. func. of t. & are related by $y = x^2 + 3$

Find $\frac{dy}{dt}$ when $x = 1$ given $\frac{dx}{dt} = 2$ when $x = 1$

Given $x = 1$, $\frac{dx}{dt} = 2$ Find $\frac{dy}{dt}$, $y = x^2 + 3$

$\frac{d}{dt} \rightarrow \frac{dy}{dt} = 2x\frac{dx}{dt} = 2(1)(2) \boxed{= 4}$

Ex 2: Air into spherical balloon @ rate of 4.5 cubic ft. per min.
Find rate of change of the radius when radius is 2 feet



Given: $\frac{dV}{dt} = {}^{+}4.5\ ft^3/min$ $\quad r = 2\ ft$

find $\frac{dr}{dt}$ $\quad$ Use: $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = \frac{4\pi}{3}\left(3r^2\frac{dr}{dt}\right) \rightarrow 4\pi r^2\frac{dr}{dt}$

$4.5 = 4\pi(2)^2\frac{dr}{dt} \rightarrow \frac{4.5}{16\pi} = \frac{dr}{dt}$

$\rightarrow 0.9 = \frac{dr}{dt} \rightarrow$ The radius is changing @ 0.9 ft/min when $r = 2$

Unit 2 Review

① $f(x)=12$ $\frac{dy}{dx}=\lim_{h\to0}\frac{f(x+h)-f(x)}{h} \to \frac{-12}{h} \to 0$

③ $f(x)=x^2-4x+5 \to \lim_{h\to0}\frac{(x+h)^2-4(x+h)+5-(x^2-4x+5)}{h}$

$\to \frac{\cancel{x^2}+2xh+h^2-\cancel{4x}-4h+\cancel{5}-\cancel{x^2}+\cancel{4x}-\cancel{5}}{h} \to \frac{2xh+h^2-4h}{h} \to \frac{h(2x+h-4)}{h}$

$\to \lim_{h\to0} 2x+h-4 \to 2x-4$

Warm up: Find $\frac{d^2y}{dx^2}$ of $4x^2+2y^2=8$

$\to 8x+4y\frac{dy}{dx}=0 \to \frac{dy}{dx}=\frac{-8x}{4y} \to \frac{-2x}{y} \to \frac{d^2y}{dx^2}=$

$\to \frac{y(-2)+(2x)\left(\frac{dy}{dx}\right)}{y^2} \to \frac{-2y+2x\frac{dy}{dx}}{y^2} \to \frac{-2y+2x\left(\frac{-2x}{y}\right)}{y^2}$

$\frac{\frac{-2y}{1}\cdot\frac{y}{y}+2x\left(\frac{-2x}{y}\right)}{y^2} \to \frac{\frac{-2y^2}{y}+\frac{-4x^2}{y}}{y^2} \to \frac{-2y^2-4x^2}{y}\cdot\frac{1}{y^2}$

$=\boxed{\frac{-1(2y^2+4x^2)}{y^3}=\frac{d^2y}{dx^2}}$

Practice trig derivatives!

d/dx sinx = cosx   d/dx cosx = −sinx   d/dx tanx = sec²x

d/dx cotx = −csc²x   d/dx secx = secxtanx   d/dx csc = −cscxcotx

d/dx cosx = −sinx   d/dx sinx = cosx   d/dx secx = secxtanx

d/dx tanx = sec²x   d/dx cot x = −csc²x   d/dx cscx = −cscxcotx

$\cos^2x+\sin^2x=1$ $1+\tan^2x=\sec^2x$ $1+\cot^2x=\csc^2x$

sin = cosx   cosx = −sinx   secx = secxtanx

cscx = −cscxcotx   tanx = sec²x   cot = −csc²x

d/dx sin = cosx   d/dx cosx = −sinx   d/dx secx = secxtanx

d/dx cscx = −cscxcotx $\frac{d}{dx}$ tanx = sec²x $\frac{d}{dx}$ cotx = −csc²x

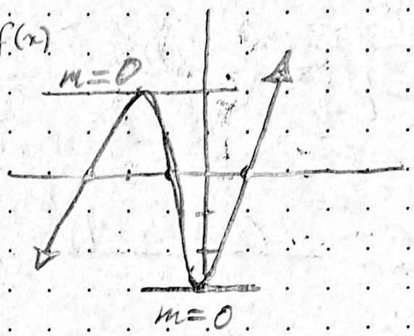
pyth

$\sin^2x+\cos^2x=1$ $1+\tan^2x=\sec^2x$ $1+\cot^2x=\csc^2x$

3.1 Extrema on an Interval Oct 22 2024

Warm-up: relative extrema of $f(x) = (x+3)(x^2-1) \to x^3 - x + 3x^2 - 3 = f(x)$

Maximum: $x = -2.155$ Minimum: $x = 0.155$
$y = 3.079$ $y = -3.079$



There is a relative maximum of 3.079 at $x \approx -2.155$.
There is a relative minimum of -3.079 at $x \approx 0.155$.

$m = f'(x) = (x+3)(2x) + (x^2-1)(1)$

x-vals: max: -2.155 min: 0.155

$(-\infty, -2.155)$ $(-2.155, 0.155)$ $(0.155, \infty)$ $\quad f'(x) = (x+3)(2x) + (x^2-1)$

f' $\quad -3$ $(+)$ $\quad 0$ $(-)$ $\quad 2$ $(+)$

f

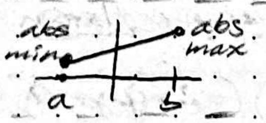
3.1 find relative & absolute extrema

Definition of Extrema → Let f be on interval I, containing c.

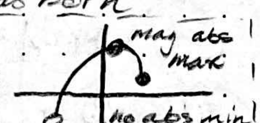
1. $f(c)$ is the minimum of f on I when $f(c) \le f(x)$ for all x on I
2. $f(c)$ is the maximum of f on I when $f(c) \ge f(x)$ for all x on I

Extreme Value Theorem Absolute max & min guarateed on closed interval

If f is continuous on a closed interval $[a, b]$ then f has both minimum & maximum on interval



$[a, b]$:



Definition of Relative Extrema

1: Open interval containing "c" on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of "f" or f has relative maximum @ $(c, f(c))$.

2. " is a minimum "
relative minimum " relative minimum @ $(c, f(c))$.

Relative = local Absolute = global

Definition of Critical Number

If $f'(c) = 0$ or "f" doesn't have $f'(c)$, then c is a critical num. of "f"

Critical numbers are potential mins. or maxs. of "f"

Include vert. asymptotes, x-vals. of holes on critical number list

Relative Extrama Occur Only @ critical numbers

Vertical asymptotes & holes aren't Extrema

Ex 2: Find Extrema of $f(x) = 2\sin x - \cos 2x$ on $[0, 2\pi]$ abs plugin end points

find f' → $f'(x) = 2\cos x + \sin 2x(2)$ → $2\cos x + 2\sin 2x = f'(x)$

$\sin 2x = 2\cos x + 2(2\sin x \cos x)$

$'' \quad = 2\cos x(1 + 2\sin x)$

$'' \quad = 2\cos x \qquad 1 + 2\sin x = 0$

$\sin x = -\frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

$f(x) = 2\sin x - \cos 2x$

$x = \frac{\pi}{2} \quad 2(1) - (-1) \to 3$

$x = \frac{3\pi}{2} \quad 2(-1) - (-1) \to -1$

$x = \frac{7\pi}{6} \quad 2(-\frac{1}{2}) - (-\frac{1}{2}) \to -1.5$

$x = \frac{11\pi}{6} \quad 2(-\frac{1}{2}) - (\frac{1}{2}) \to -1.5$

$x = 2\pi \quad (2(0)) - (1)) \to -1$

Plug into original

x	0	$\pi/2$	$\frac{3\pi}{2}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	2π
$f(x)$	0	3	-1	-1.5	-1.5	-1

abs max = 3 @ $x = \pi/2$

abs min $x = -3/2$ @ $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

How to find absolute Extrema on [a, b]

1. find $f'(x)$ 2. Set $f'(x)$ = to "0" & find critical values

3. Plug in all critical values & end points into $f(x)$

4. Interpret results ⇒ highest y-value is the abs max & lowest is abs min

How to find relative Extrema

1. find $f'(x)$ 2. Set $f'(x)$ equal to "0" & find critical values

3. Make intervals 4. Select test point from every interval & plug it into $f'(x)$ - derivative

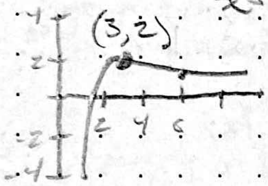
5. If there is a sign change @ critical value, that c.v. is a location of a relative extrema

Change from (+) to (−) is rel. max.
from (−) to (+) is rel. min.

no change = no extrema

Ex 1: Derivative @ Rel. Extrema

(a) $f(x) = \frac{9(x^2-3)}{x^3}$ $\quad f'(x) = \frac{x^3(9(x^2-3)(2x)) - (9(x^2-3))(3x^2)}{x^6}$



$f'(3)=0$

$(m @ x=3) = 0$

maximum of 2 @ $x=3$.

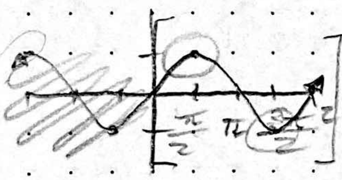
(b) $f(x) = |x|$

minimum of "0" @ $x=0$

no slope → cusp / sharp curve

not differentiable @ $x=0$

(c) $f(x) = \sin x$ on $[0, 2\pi]$ $\qquad f'(x) = \cos x$



max of 1 @ $x = \frac{\pi}{2}$ $\qquad f'(x) = 0$ @ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

min of -1 @ $x = \frac{3\pi}{2}$

3.3 Increasing & Decreasing Functions & 1st $\frac{dy}{dx}$ test (Oct. 24. 2024)

Definition of Increasing & Decreasing Functions:

A function is increasing on the interval x_1 & x_2 if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function is decreasing on x_1 & x_2 if $x_1 < x_2$, implies $f(x_1) > f(x_2)$

Test for increasing / decreasing functions:

if on interval $[a, b]$ continuous & differentiable on (a, b):

1: if $f'(x) > 0$ for all x on (a, b), f is increasing on $[a, b]$

2: if $f'(x) < 0$ for all x on (a, b), f is decreasing on $[a, b]$

3: if $f'(x) = 0$ for all x on (a, b), f is constant on $[a, b]$

Ex 1:

$f(x) = x^3 - 4x$ $\quad f'(x) = 3x^2 - 4$ $\qquad 0 = 3x^2 - 4 \rightarrow 4 = 3x^2$

$\rightarrow \frac{4}{3} = x^2 \rightarrow x = \pm\sqrt{\frac{4}{3}} \rightarrow x = \pm\frac{2}{\sqrt{3}}$

$(-\infty, -\frac{2}{\sqrt{3}})\ (-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})\ (\frac{2}{\sqrt{3}}, \infty)$ | $f(x)$ is increasing on $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$.

$-3 \qquad 0 \qquad 3$

f': $+ \qquad - \qquad +$ | $f(x)$ is decreasing on $(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}})$